Dirac Equation

Elastic collisions of leptons and quarks proceed by electromagnetic or weak interactions. Since leptons and quarks are spin-1/2 fermions, the appropriate theoretical framework is the Dirac equation.

In this lecture I shall sketch the steps that lead from the basic principles of quantum mechanics to the Dirac equation and further outline the case of elastic collisions of two spin-1/2 fermions by e.m. interaction.

Elastic collisions mediated by weak interactions are treated similarly.

Dirac Equation:

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi, \qquad H = \vec{\alpha}\cdot\vec{p}c + \beta mc \qquad (1)$$

Coefficients α and β satisfy the following anti-commutation relations:

$$\alpha_x\alpha_y + \alpha_y\alpha_x = \alpha_y\alpha_z + \alpha_z\alpha_y = \alpha_z\alpha_x + \alpha_x\alpha_z = 0$$

$$\alpha_i\beta + \beta\alpha_i = 0, \quad i = x, y, z$$

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1$$

Their hermiticity property is (from hermiticity of *H* and *p*)

$$\vec{\alpha}^\dagger = \vec{\alpha}, \quad \beta^\dagger = \beta$$

They can be represented by 4-by-4 matrices; in ***standard representation*** these are

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}$$

where the σ are 2-by-2 matrices (Pauli matrices):

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The wave function ψ is a 4-component spinor (bi-spinor, Dirac spinor):

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}$$

The hermitian conjugate of Eq. (1) is

$$-i\hbar \frac{\partial \psi^\dagger}{\partial t} = (H\psi)^\dagger \qquad (2)$$

To get the continuity eqn, premultiply (1) by $\psi^\dagger$ and postmultiply (2) by ψ, then subtract, hence:

$$i\hbar \frac{\partial \psi^\dagger \psi}{\partial t} = \psi^\dagger H\psi - (H\psi)^\dagger \psi = -i\hbar \nabla \cdot \left(\psi^\dagger \vec{\alpha} \psi\right)$$

define

$$\rho(x) = \psi^\dagger \psi, \quad \vec{j} = \psi^\dagger \vec{\alpha} \psi$$

hence get continuity equation:

$$\frac{\partial \rho(x)}{\partial t} + \nabla \cdot \vec{j} = 0$$

Combine ρ and vector j into four-vector:

$$j^{\mu} = \left(\rho, \vec{j}\right)$$

and define the 4D differential operator

$$\partial^{\mu} = \left(\partial_t, -\nabla\right)$$

hence

$$\boxed{\partial^{\mu} j_{\mu} = 0}$$

Covariant form of the Dirac equation:

define

$$\gamma^{0} = \beta, \quad \vec{\gamma} = \beta\vec{\alpha},$$

hence

$$\left(i\gamma^{\mu}\partial_{\mu} - \frac{mc}{\hbar}\right)\psi\left(x\right) = 0$$

or with $\hbar = 1$ and $c = 1$

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x) = 0$$

The γ matrices have the following properties:

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}$$
$$\gamma^{\mu\dagger} = \gamma^{0}\gamma^{\mu}\gamma^{0}$$
$$\mathrm{Tr}\,\gamma^{\mu} = 0$$

where $g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ is the metric tensor of special relativity.

The *adjoint equation* is

$$\bar{\psi}(x)\left(i\gamma^{\mu}\overleftarrow{\partial}_{\mu} + m\right) = 0, \quad \text{where} \quad \bar{\psi}(x) = \psi^{\dagger}\gamma^{0}$$

and $\overleftarrow{\partial}_{\mu}$ operates to the left. The 4-vector current can be shown to be

$$j^{\mu} = \bar{\psi}\gamma^{\mu}\psi$$ (*probability current density*)

Plane wave solutions of the Dirac equation:

$$\psi(x) = u(p) e^{-ip\cdot x}$$

hence

$$(\not{p} - m) u(p) = 0, \qquad \not{p} = \gamma^{\mu} p_{\mu}$$

In standard representation of the γ matrices we get

$$\not{p} = \begin{pmatrix} E & -\sigma\cdot p \\ \sigma\cdot p & -E \end{pmatrix}$$

Let

$$u(p) = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

then

$$\vec{\sigma}\cdot\vec{p}\,u_A - (E+m) u_B = 0$$

$$(E-m) u_A - \vec{\sigma}\cdot\vec{p}\,u_B = 0$$

Eliminate u_A, then u_B, hence

$$\left(E^2 - \vec{p}^2 - m^2\right) u_{A,B} = 0$$

and hence the eigenvalues:

$$E = \pm\sqrt{\vec{p}^2 + m^2}$$

The negative energy states are interpreted as antiparticle states of positive energy travelling backwards in time.

The eigenvectors are found by standard matrix algebra; for $E > 0$ we get

$$u_s(p) = N \begin{pmatrix} \varphi_s \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m}\varphi_s \end{pmatrix}, \quad s = 1,2$$

where

$$\varphi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad N = \text{normalization factor}$$

For $E < 0$:

$$u_{s+2}(p) = N \begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E-m}\varphi_s \\ \varphi_s \end{pmatrix}, \quad s = 1,2$$

Orthogonality of the eigenvectors:

$$u_r^\dagger(p)u_s(p)=0 \quad \text{if} \quad r\neq s$$

Normalization: $2E$ particles per unit volume, *i.e.*

$$\int_{\text{unit volume}} \psi^\dagger(x)\psi(x)\,dV = 2E$$

hence

$$N=\sqrt{E+m}$$

Antiparticles. Let $E<0$ and make the substitution

$$E\to -E, \quad \vec{p}\to-\vec{p}$$

in the negative energy solutions, hence

$$(-\not{p}-m)u_{s+2}(-p)=0$$

or setting $v_s(p)=u_{s+2}(-p)$

$$\boxed{(\not{p}+m)v_s(p)=0}$$

Completeness relations.

$$\sum_{s=1,2} u_s(p)\bar{u}_s(p) = \not{p} + m, \quad \sum_{s=1,2} v_s(p)\bar{v}_s(p) = \not{p} - m$$

Helicity.

We have two solutions for $E > 0$, *i.e.* we have a two-fold degeneracy, so there must be an additional degree of freedom. This additional degree of freedom is interpreted as the electron spin.

Formally the degeneracy means that there is an operator that commutes with the Hamiltonian. Such an operator is

$$\vec{\Sigma} \cdot \vec{p} = \begin{pmatrix} \vec{\sigma} \cdot \hat{p} & 0 \\ 0 & \vec{\sigma} \cdot \hat{p} \end{pmatrix}, \quad \hat{p} = \vec{p}/p$$

The operator $\frac{1}{2}\vec{\sigma} \cdot \hat{p}$ is the projection of the electron spin onto the direction of the momentum; it is called the ***helicity operator***; its eigenvalues $\lambda = \pm 1/2$ are the electron helicities.

Bilinear covariants.

We have previously seen that the expression

$$j^{\mu} = \bar{\psi}\gamma^{\mu}\psi$$

is a 4-vector. One can construct other expressions bilinear in ψ and $\bar{\psi}$

They are of the general form of $\bar{\psi}\Gamma_i\psi, \quad i = 1, 2, \ldots, 16$

The number 16 corresponds to the number of linearly independent 4-by-4 matrices. In relativistic quantum mechanics one proves that the bilinear forms one can construct have the transformation properties shown in the following table:

matrix Γ_i	1	γ^{μ}	$\sigma^{\mu\nu}$	$\gamma^5\gamma^{\mu}$	γ^5
$\bar{\psi}\Gamma_i\psi$	S	V	T	A	P

where $\sigma^{\mu\nu} = \frac{1}{2}\left(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}\right), \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and

S = Scalar, V = Vector, T = Tensor, A = Axial vector, and P = Pseudo scalar.

The γ^5 matrix has the following properties:

$$\gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0$$

$$\gamma^{5\dagger} = \gamma^5, \quad \left(\gamma^5\right)^2 = 1$$

In standard representation we have

$$\gamma^5 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}$$

Electrodynamics of spin-1/2 fermions

The Dirac eqn of a spin-1/2 fermion of charge ***e*** in an electromagnetic field is

$$(\not{p} + e\not{A} - m)\psi(x) = 0$$

where A^μ is the e.m. 4-vector potential. We can rewrite this in the form of

$$(\not{p} - m)\psi(x) = \gamma^0 V \psi(x), \quad V = -e\gamma^0 \not{A}$$

Here the interaction potential ***V*** is chosen such as to be consistent with the corresponding expression in the nonrelativistic limit.

In perturbation theory one learns that the transition amplitude from initial state ψ_i to final state ψ_f is in the lowest order given by

$$T_{fi} = -i\int \psi_f^\dagger V \psi_i \, d^4x = ie\int \bar{\psi}_f \not{A} \psi_i \, d^4x = -i\int j^\mu A_\mu \, d^4x$$

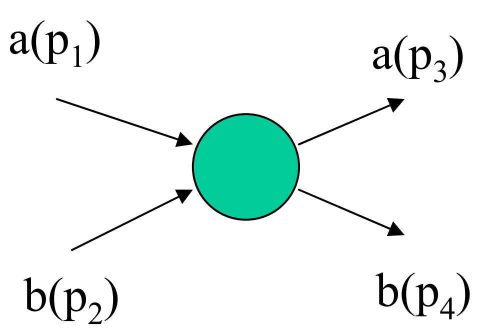
$$j^\mu = -e\bar{\psi}_f \gamma^\mu \psi_i$$ ***transition current*****.**

Note the factor of -*e* (with *e* >0) to make this an electric current density!

Now we want to apply this formalism to the collision of two charged particles **a** and **b**. Particle **a** is described by the transition current j_a^μ, and particle **b** is the source of the e.m. field A^μ. From Maxwell's theory the wave equation of A^μ with a source term j^μ is

$$\partial^\nu \partial_\nu A^\mu = j_b^\mu$$

Let the 4-momentum of particle **b** before (after) emitting a photon be p_2 (p_4). Then the corresponding wave functions are



$$\psi_{bi} = u_2 e^{-ip_2 \cdot x}, \quad \overline{\psi}_{bf} = \overline{u}_4 e^{ip_4 \cdot x}$$

$$j_b^\mu = -e\left(\overline{u}_4 \gamma^\mu u_2\right) e^{iq\cdot x}; \quad q = p_4 - p_2$$

$$A^\mu = \left(\partial^\nu \partial_\nu\right)^{-1} j_b^\mu = -\frac{1}{q^2} j_b^\mu$$

$$T_{fi} = -i\int d^4x\, j^\mu(a)\left(-\frac{1}{q^2}\right) j_\mu(b)$$

After integration over x we have

$$T_{fi} = -i(2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) M$$

where

$$-iM = \left[i j^\mu(a) \right] \left[-\frac{ig_{\mu\nu}}{q^2} \right] \left[i j^\nu(b) \right]$$

$$|M|^2 = \frac{1}{q^4} \left(j^\mu(a) j_\mu(b) \right)^* \left(j^\nu(a) j_\nu(b) \right) = \frac{1}{q^4} \left[j^{\mu*}(a) j^\nu(a) \right] \left[j^*_\mu(b) j_\nu(b) \right]$$

The expressions in square brackets are tensors. Consider first one:

$$L^{\mu\nu} = j^{\mu*}(a) j^\nu(a) = e^2 \left(\overline{u}_3 \gamma^\mu u_1 \right)^\dagger \left(u_3 \gamma^\nu u_1 \right) = e^2 \left(u_1^\dagger \gamma^{\mu\dagger} \overline{u}_3^\dagger \right) \left(\overline{u}_3 \gamma^\nu u_1 \right)$$

Recall:

$$\overline{u}^\dagger = \left(u^\dagger \gamma^0 \right)^\dagger = \gamma^0 u; \quad \gamma^\mu = \gamma^0 \gamma^{\mu\dagger} \gamma^0$$

hence

$$L^{\mu\nu} = e^2 \sum_{klmn} \left(\overline{u}_{1k} \gamma^\mu_{kl} u_{3l} \right) \left(\overline{u}_{3m} \gamma^\nu_{mn} u_{1n} \right)$$

where k, l, m and n label the matrix elements.

We can move u_{1n} to the front!

$$L^{\mu\nu} = e^2 \sum_{klmn} \left(\bar{u}_{1k}\gamma^{\mu}_{kl}u_{3l}\right)\left(\bar{u}_{3m}\gamma^{\nu}_{mn}u_{1n}\right)$$

hence

$$L^{\mu\nu} = e^2 \mathrm{Tr}\, u_1\bar{u}_1\gamma^{\mu}u_3\bar{u}_3\gamma^{\nu}$$

The spin labels have not been written, but we must remember the spins of the electron before and after the collision: s_1 and s_3

If the spin is not observed, then we must carry out a ***spin summation***: we must ***average*** over the initial and ***sum*** over the final spin:

$$\bar{L}^{\mu\nu} = \frac{1}{2}e^2 \sum_{s_1 s_3} \mathrm{Tr}\, u_1\bar{u}_1\gamma^{\mu}u_3\bar{u}_3\gamma^{\nu} = \frac{1}{2}e^2 \mathrm{Tr}\left(\sum_{s_1} u_1\bar{u}_1\right)\gamma^{\mu}\left(\sum_{s_3} u_3\bar{u}_3\right)\gamma^{\nu}$$

$$= \frac{1}{2}e^2 \mathrm{Tr}(\not{p}_1 + m)\gamma^{\mu}(\not{p}_3 + m)\gamma^{\nu}$$

and finally

$$\bar{L}^{\mu\nu} = \frac{1}{2}e^2 \mathrm{Tr}\left[\not{p}_1 \gamma^\mu \not{p}_3 \gamma^\nu + m\left(\gamma^\mu \not{p}_3 \gamma^\nu + \not{p}_1 \gamma^\mu \gamma^\nu\right) + m^2 \gamma^\mu \gamma^\nu \right]$$

<u>Trace theorems:</u>

$$\mathrm{Tr}\ \gamma^\mu = 0$$

$$\mathrm{Tr}\ \gamma^\mu \gamma^\nu = 4g^{\mu\nu}$$

$$\mathrm{Tr}\ \gamma^\mu \gamma^\nu \ldots \gamma^\omega = 0 \quad \textit{for an odd number of factors}$$

$$\mathrm{Tr}\ \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho = 4\left(g^{\mu\nu} g^{\lambda\rho} - g^{\mu\lambda} g^{\nu\rho} + g^{\mu\rho} g^{\nu\lambda}\right)$$

<u>Completion of calculation of the electron tensor</u>: using the trace theorems we get

$$\bar{L}^{\mu\nu} = 2e^2 \left[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu + \left(m^2 - p_1 \cdot p_3\right) g^{\mu\nu} \right]$$

but $\quad q^2 = \left(p_1 - p_3\right)^2 = 2m^2 - 2p_1 \cdot p_3$

hence

$$\boxed{\bar{L}^{\mu\nu} = 2e^2 \left[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu + \frac{1}{2} q^2 g^{\mu\nu} \right]}$$

Recall: the mod-square of the transition matrix was

$$|M|^2 = \frac{1}{q^4}\left[j^{\mu *}(a)\, j^{\nu}(a) \right]\left[j^{*}_{\mu}(b)\, j_{\nu}(b) \right]$$

or after spin summation

$$|\bar{M}|^2 = \frac{1}{q^4} \bar{L}_a^{\mu\nu} \bar{L}^b_{\mu\nu}$$

and after contraction over the Lorentz labels we get

$$|\bar{M}|^2 = \frac{8e^4}{q^4}\left(p_1 \cdot p_2 p_3 \cdot p_4 + p_1 \cdot p_4 p_3 \cdot p_2 - m_a^2 p_2 \cdot p_4 - m_b^2 p_1 \cdot p_3 + m_a^2 m_b^2 \right)$$

<u>Mandelstam variables</u> (invariants):

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2$$

by 4-momentum conservation we have

$$s + t + u = 2m_a^2 + 2m_b^2$$

Our calculation was done in preparation of elastic electron-quark scattering at high energy, where the masses can be neglected. This gives us finally the spin-averaged mod-squared transition matrix in the following form:

$$\left|\bar{M}\right|^2 = \frac{4e^4}{t^2}\left(s^2 + u^2\right)$$

and hence the differential cross section:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{s}\frac{s^2 + u^2}{t^2}$$