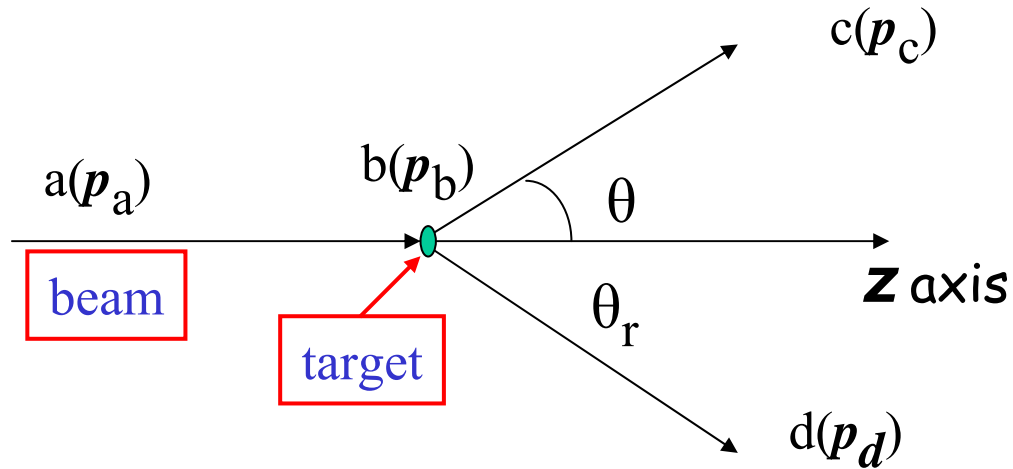


# High Energy Physics

## **Lecture 3:** Kinematics of Particle Reactions



LAB kinematics diagram of particle collision  $a + b \rightarrow c + d$

**Definition of LAB frame:** the target particle is at rest

The particles have masses  $m_a, m_b, m_c, m_d$

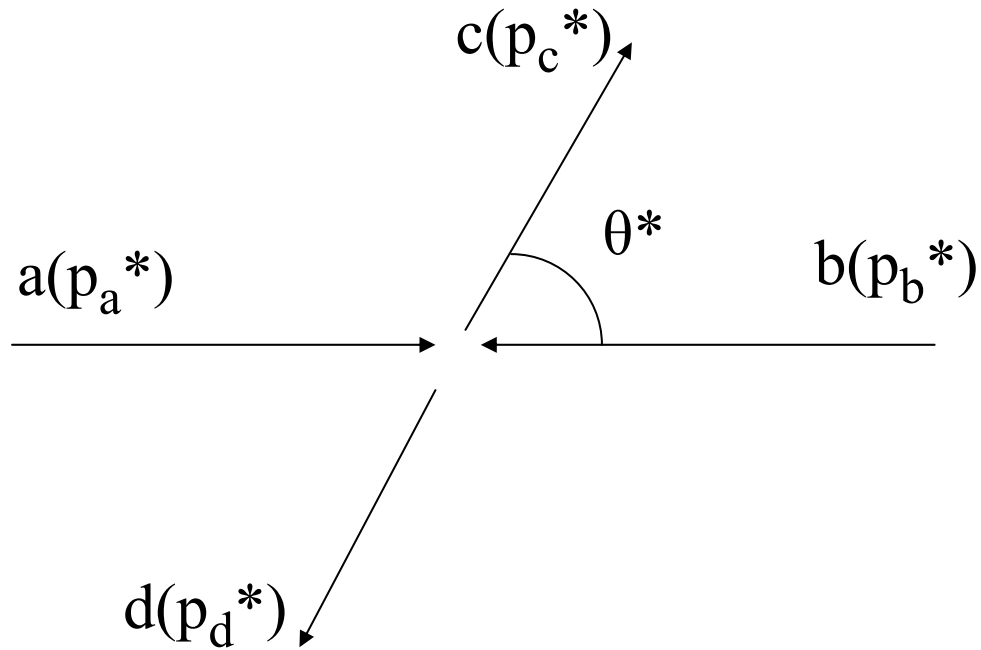
Particle  $a$  is the **incident** particle

Particle  $b$  is the **target** particle: this is initially at rest in the LAB

Particle  $c$  is the **scattered** particle

Particle  $d$  is the **recoil** particle

$\theta$  is the LAB scattering angle;  $\theta_r$  is the recoil angle



Kinematics diagram of particle collision in the CMS

Definition of CMS:

the total momentum of the initial system is zero.

It follows by momentum conservation that the total momentum of the final system is also equal to zero.

## Energy and momentum conservation:

$$E_a + E_b = E_c + E_d$$

$$\vec{p}_a + \vec{p}_b = \vec{p}_c + \vec{p}_d \quad (\text{vectors!})$$

$E = \sqrt{(mc^2)^2 + (pc)^2}$  is the total relativistic energy

$c = \text{speed of light}$  (in the vacuum!)

In particle physics one frequently uses units such that

$$c = 1 \quad (\text{and} \quad \hbar = 1 \quad )$$

and then the energy-momentum relation is

$$E = \sqrt{m^2 + p^2}$$

**Elastic scattering:**  $m_a = m_c, m_b = m_d$

Typical problem of particle kinematics  
(needed by experimentalists):

Given the masses of the initial particles and the momentum of the incident particle, find the momenta of the final particles in an elastic collision;  
also find the LAB recoil angle.

The problem is solved by using energy and momentum conservation:

$$p_c = p \frac{(s + m_a^2 - m_b^2) \cos \theta \pm 2W \sqrt{m_b^2 - m_a^2 \sin^2 \theta}}{2(s + p^2 \sin^2 \theta)}$$

where  $p$  is the LAB momentum of  $a$ ,

$W = E_{LAB} + m_b$  is the total LAB energy of the initial system,

$$s = m_a^2 + m_b^2 + 2m_b E_{LAB}, \quad p_d = \sqrt{p^2 + p_c^2 - 2pp_c \cos \theta}$$

$$\tan \theta_r = p_c \sin \theta / (p - p_c \cos \theta)$$

Example: elastic pion - proton scattering

$$m_{\pi} = 140 \text{ MeV}, \quad m_p = 940 \text{ MeV},$$

Assume: LAB energy of the incident pion  $E = 1000 \text{ MeV}$

Let  $\theta = 30 \text{ deg}$

Question: what are the momenta and energies of the scattered pion and of the recoil proton; what is the recoil angle?

Using the above formulas we get the following answer:

$$\begin{aligned} p_{\pi} &= 866.3 \text{ MeV}, & E_{\pi} &= 877.54 \text{ MeV} \\ p_p &= 495.2 \text{ MeV}, & E_p &= 1062.46 \text{ MeV} \\ \theta_r &= 1.07 \text{ rad} = 61 \text{ deg}. \end{aligned}$$

Note the conservation of energy: initially we had a total energy

$$W = E + m_p = 1940 \text{ MeV};$$

after the collision we have  $E_{\pi} + E_p = 877.54 + 1062.46 = 1940 \text{ MeV}$

Our result looks surprising: the proton, which was initially at rest and which was hit by a pion of LAB energy 1000 MeV, has acquired an energy of 1062 MeV!

The reason is that these energies are *relativistic total energies*: they include the rest energy, which in the case of the proton is 940 MeV.

More intuitive than the total energy is the kinetic energy (K.E.): this is defined by **K.E. = total energy - rest energy**:

$$T = E - mc^2$$

We can check that the expression for  $T$  takes on the familiar form of the nonrelativistic K.E. if the particle velocity is small compared with  $c$ :

$$T = \sqrt{m^2 c^4 + p^2 c^2} - mc^2 = mc^2 \sqrt{1 + p^2 / m^2 c^2} - mc^2$$

and since nonrelativistically  $p / mc \ll 1$  we have

$$\sqrt{1 + p^2 / m^2 c^2} \approx 1 + p^2 / 2m^2 c^2 \quad \text{hence} \quad T_{nr} \approx \frac{p^2}{2m}$$

In our example of elastic pion – proton scattering, the energy balance, expressed in terms of the K.E.s, is

$$(T_{\pi} + m_{\pi}c^2) + (T_p + m_p c^2) = (T'_{\pi} + m_{\pi}c^2) + (T'_p + m_p c^2)$$

*i.e.* the rest energies cancel and we are left with the balance of K.E.s:

$$T_{\pi} + T_p = T'_{\pi} + T'_p$$

and in our previous numerical example we have:

$$T_{\pi} = 860 \text{ MeV}, T_p = 0, \quad T'_{\pi} = 737.54 \text{ MeV}, T'_p = 112.46 \text{ MeV}$$

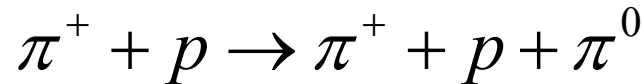
and we see that only a small part of the initial pion K.E. got transferred to the proton.

But remember: the K.E.s balance only in the case of elastic scattering. In *inelastic* collisions only the *total* energies balance!



## Inelastic collisions.

Example of an inelastic collision:



Here an additional (neutral) pion has been created.

A first question is: what LAB K.E. is needed to produce this extra pion?  
(The smallest energy to produce the extra pion is called *threshold energy*)

The calculation is simple in the CMS.

To find the LAB threshold K.E. we then have two ways:

- 1) do a Lorentz transformation from CMS to LAB; this is difficult  
or
- 2) use the concept of invariance: that's the easy way

Let us do the calculation in the CMS, then use invariance.

## CMS threshold energy.

By definition of the CMS, the total momentum is equal to zero, both before and after the reaction.

Before the reaction we have

$$\vec{p}_\pi + \vec{p}_p = 0, \quad \text{hence} \quad p_\pi = p_p$$

and we can drop the subscripts on the momenta;  
the total CMS energy before the collision is therefore

$$E_{in} = \sqrt{m_\pi^2 + p^2} + \sqrt{m_p^2 + p^2} \quad \boxed{\text{("in" for initial)}}$$

Usually the square of  $E_{in}$  is denoted by  $s$ , *i.e.* we have

$$\sqrt{s} = \sqrt{m_\pi^2 + p^2} + \sqrt{m_p^2 + p^2}$$

and if we solve for  $p$  (**Exercise!**), then we get

$$p_{CMS} = \sqrt{\left[ s - (m_\pi - m_p)^2 \right] \left[ s - (m_\pi + m_p)^2 \right]} / 2\sqrt{s}$$

After the reaction we have in the CMS

$$s = \left\{ \sqrt{m_1^2 + p_1^2} + \sqrt{m_2^2 + p_2^2} + \sqrt{m_3^2 + p_3^2} \right\}^2$$

and the minimum of this corresponds to *all* final state particles being at rest in the CMS, *i.e.*  $\mathbf{p}_i = 0$ ,  $i=1,2,3$ , hence

$$s_{\min} = M^2 = \{m_1 + m_2 + m_3\}^2$$

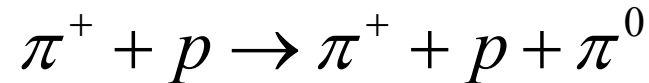
One can show that the quantity  $s$  is given in terms of the LAB energy of the incident particle by the following expression:

$$s = m_\pi^2 + m_p^2 + 2m_p E_{LAB}$$

and this is *invariant*, meaning that it takes the same value in any reference frame. We can therefore equate the threshold value of  $s$  in the LAB with its threshold value in the CMS, hence

$$E_{LAB}^{thr} = \frac{1}{2m_p} (M^2 - m_\pi^2 - m_p^2)$$

Putting in the numbers for our example of production of a neutral pion in a pion-proton collision:



$$m_{\pi^+} = 140 \text{ MeV}, \quad m_{\pi^0} = 135 \text{ MeV}, \quad m_p = 940 \text{ MeV}$$

we get (Exercise!):  $M = 1215 \text{ MeV}$

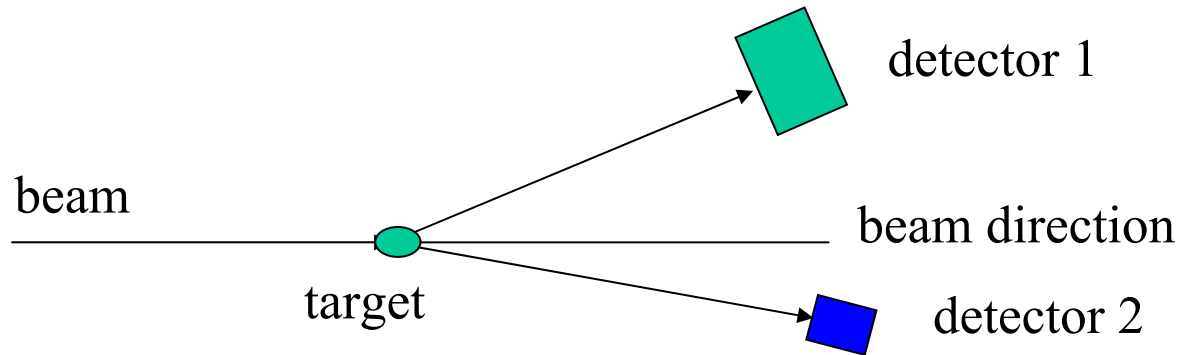
and for the LAB energy and LAB K.E. of the incident pion (Exercise!)

$$E_{LAB}^{thr} = 305 \text{ MeV}, \quad T_{LAB}^{thr} = 165 \text{ MeV}$$

Thus the LAB threshold K.E. is greater than the extra mass produced. This is so because in the LAB the entire final state system is moving, *i.e.* it has a K.E. which is taken from the K.E. of the incident particle.

## Use of kinematics in planning an experiment:

In a 2-to-2 reaction, the final momenta span a plane, called the **reaction plane**. By momentum conservation the reaction plane contains the beam axis (**z axis!**). We can therefore position two detectors such as to detect the two final state particles:



We can do something a bit more clever than just position the detectors at the angles expected for the reaction we want to study: we know also the speed of the particles and therefore the time it takes them to get to the detectors: **Time of Flight (ToF)**. So we put the detectors at such distances that the ToFs coincide, then use electronics to count only particles which arrive at the detectors in **coincidence**.

## Time of Flight calculation

Distance between interaction point (target!) and detector:  $D$

Particle velocity .....:  $v$

hence ToF .....:  $t = D/v$

*But velocities are relativistic in typical particle physics experiments!*

The relativistic relation between momentum and velocity is

$$p = \gamma m v$$

where  $\gamma = 1/\sqrt{1 - (v/c)^2}$  (“relativistic  $\gamma$  factor”)

The relativistic relation for the energy is

$$E = \gamma m c^2$$

hence  $v/c = pc/E$

this is the particle velocity “in units of  $c$ ”; one says for instance:

“the particle is travelling at a speed of 0.9 of the speed of light”.

Example:  $\pi^+ p \rightarrow \pi^+ p$  with  $E_{LAB} = 1000 \text{ MeV}$

Note that we have plenty of energy to produce additional pions:  
to produce just a single (neutral) pion we need only 305 MeV;  
you can work out for yourselves how many pions can be produced  
if the beam energy is 1000 MeV (**Exercise!**).

So the design of our experiment should be such as to count all  
elastic collisions and not to count the inelastic ones.

We have previously worked out for a scattering angle of 30 deg:

$$\begin{aligned} p_\pi &= 866.3 \text{ MeV}/c; & E_\pi &= 877.54 \text{ MeV} \\ p_p &= 495.2 \text{ MeV}/c; & E_p &= 1062.46 \text{ MeV} \quad \text{and} \quad \theta_r = 61 \text{ deg.} \end{aligned}$$

Hence  $v_\pi = 866.3/877.54 = 0.987 c$  and  $v_p = 0.466 c$

If we put the pion detector at a distance  $D_\pi = 1$  m from the target, then the ToF of the pion is

$$t_\pi = \frac{1\text{m}}{0.987c} \approx 0.34\text{ns}$$

For coincidence the proton ToF must be equal to the pion ToF; therefore the distance at which we must put the proton detector is

$$D_p = v_p \cdot t_\pi = D_\pi \cdot \frac{v_p}{v_\pi} = \frac{0.466}{0.987} = 0.47\text{m}$$

Please remember to do all indicated exercises;  
also check my numbers: I may have had butterfingers  
and got the numbers wrong!